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THE BOUNDARY PROBLEM FOR SOME CLASSES OF WAVE EQUATIONS IN NONSMOOTH DOMAINS

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INTRODUCTION

1. History and significance of the problem

The study of the linear boundary value problems for partial differential equations, systems in the domain with smooth boundary so far have touched nearly every aspects of the issue in the first half of 20th century. G. Fichera (1972), D. Ginbarg and N. Trudinger (1983), L. C. Evans (1998) used unit partition method to study the stop boundary value problems in smooth domains in order to turn them into full space and semi-space ones. Non-stop boundary value problems in the cylinders with the smooth bases were based on Laplace or Fourier transformation to turn the stop problem with parameters in smooth domains.

From the middle of 20th century, general boundary value problems for elliptic equations in singular boundary domains were studied, V. A. Kondrat'ev (1977, 1998) obtained important results of well-posed, regularity and asymptotic behaviour of solutions in the domain containing conical points on boundary. Later, some mathematicians such as P. Grisvard (1985), M. Dauge (1988), E. V. Frolove (1994), V. A. Kozlov, V. G. Maz'ya, J. Rossmann (1997, 2001), V. A. Kozlov, V. G. Maz'ya (1988) based on V. A. Kondrat'ev's findings to study boundary value problems for stop systems in the domains with singular points on the boundary.

In the 1990s, Fourier or Laplace transformation methods were not strong enough for the mathematicians to conclude important properties of the solutions to non-stop boundary value problems in domains with singular points on base. At the end of 20th century, PhD.Sci. Nguyen Manh Hung et al. used tangent surface cutting method to turn non-stop problems into stop ones on a tangent surface. By that method, non-stop problems with coefficient in time variable were studied in the aspects of their well-positioned in any domain and asymptotic behaviour of the solutions near conical points on the boundary. The results of weak solvability, smoothness of weak solution in time variable for the first and second initial linear problems in cylinder with any boundary base, conical and angle point contained base were obtained. Subsequently, asymptotic behaviour of the solutions near conical points of general boundary value problems for hyperbolic systems in cylinders with conical point containing bases was archieved as well. However, the non- stop problems' results were only considered in limited cylinders with non-smooth base and asymptotic behaviour of solutions near singular points in cylin-

ders with non-smooth point containing bases. Nguyen Manh Hung et al. (2004, 2005) and Vu Trong Luong et al. (2011) studied non-stop parabolic problems in unlimited cylinders. The regularity of solutions obtained from general boundary value problems in unlimited with conical point containing bases was mentioned when Nguyen Manh Hung and Nguyen Thanh Anh (2008) considered initial boundary value problems for non-stop secondary parabolic systems in limited cylinders with polygonal bases.

Basing on V.A. Kondratiev, V.G. Maz'ya and B.A. Plamenevskii's important results of initial boundary value problems for elliptic equations in non-smooth cylindrical domains such as conical point containing domains and polygonal based ones, Nguyen Manh Hung et al. obtained the uniqueness, regularity and asymptotic behaviour of solutions near singular points (conical and cuspidal points). Especially, the smoothness of generalized solutions in time variable of the first initial boundary value problems for high order hyperbolic equations in unlimited cylinders (Nguyen Manh Hung, 2007) as well as the uniqueness of generalized solutions of the first mixed problems for high order hyperbolic equations in unlimited smooth domains (Bui Trong Kim, 2008) were confirmed.

The nonlinear wave equations for structural damping on any smooth domains, in which, the elastic systems for structural damping were studied by many scientists interested in the elastic systems for structural damping of about 40 years. Fan, Li, Chen (2013) obtained the existence of mild solution in Banach spaces with damping constant $\rho \geq 2$ and nonlinear f satisfying Lipschitz condition in second variable. Fan and Gao (2014) investigated asymptotic behaviour of mild solution of elastic systems for structural damping in Banach spaces.

The results obtained have raised such questions as whether the problems are solvable or not if nonlinear terms are added to semi hyperbolic equations in unlimited non-smooth cylinders; or whether there exist solvability, regularity of solutions in time variable of initial non-boundary value problems if there was a side domain instead of those with conical, cuspidal and angle points. In another aspect, are the ways to solve initial boundary linear problems for hyperbolic in domains with edges different from those to the ones containing conical and cuspidal points. Furthermore, when studying the elastic systems for structural damping, we have found that the results for the existence of decay mild solution to the problem such as obtained for classes of nonlinear functions which have Lipschitz properties, the operators in the equation are considered on the smooth domains.

From the above issues, we decided to research on boundary problem for

some classes of wave equations in nonsmooth domains. In which, we investigate the existence global weak solution, local weak solution of the initial boundary value problems for semilinear hyperbolic equations in nonsmooth cylinders, in domains with edges. Furthermore, we also study decay mild solution at exponential speed of the nonlocal initial value problem for nonlinear second-order differential equations in Banach spaces with structural damping on smooth and nonsmooth domains.

2. Overview of the research problem

As mentioned above, the study of well-posed, regularity in time and space variable, asymptotic behavior of solution of initial boundary value for nonlinear hyperbolic equations in nonsmooth domains in general and in the infinite nonsmooth cylinders, domain with edges in particular is a the, which has been interested by many mathematicians recently. Parallel with this issues, study existence and properties of mild solution of Fractional integral and differential equations for structural damping, ect. based on results of analytics raised a high attention. One of the typical result that is existence or nonexistence of global solution is investigated J. L. Lions (1969), D. H. Sattinger (1968, 1975), H. A. Levine (1974), M. Can (1997), B. Zheng (2004). J. L. Lions concluded the existence of the solution by applying compact method and in any finite cylinders and used compact method and Faedo-Galerkin Technique. In the equation of the problem, he considered operator $L = \Delta$ and nonlinear element $f(x,t,u) = |u|^p u$.

From results obtained from V. A. Kondrat'ev, V. A. Konzlov, V. Maz'ya, J. Rossmann for elliptic equations in nonsmooth domains: domain with conical or angular points, piecewise smooth domain, convex domain, cone domain, polyhedral domain, ect. Using the results of spectrum problem associated with general elliptic equations in domains with edges, the existence uniqueness, regularity of generalized solution were investigated in weighted Sobolev L_2-space, in weighted Hölder space. Nguyen Manh Hung obtained some important results of regularity of solution in space variable of hyperbolic equation of second order

$$\begin{aligned}
&\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big) + a(x,t) - u_{tt}(t) = f(x,t), \quad (x,t) \in \Omega \times (0,\infty),\\
&u(x,0) = 0, \quad u_t(x,0) = 0, && x \in \Omega,\\
&u(x,t) = 0, && (x,t) \in \partial\Omega \times (0,\infty)
\end{aligned}$$

in infinite cylinders with nonsmooth base (2007); By using approximate Galerkin, the unique solvability of solution of mixed problem for strong hyperbolic systems of higher order with element on the right side is function $f(x,t)$

in infinite nonsmooth cylinder was obtained; By using approximate boundary method, the existence unique and regularity of solution in time variable of boundary value problem for hyperbolic system in nonsmooth cylinder whose base is a cusp domain (2006, 2008); By applying approximate Galerkin, help of elliptic problem on domains with conical points, the unique solvability of solution, the regularity, asymptotic behavior of solution near the conical points were obtained for initial boundary value problem for hyperbolic of higher order (2013)

$$\begin{aligned}
&u_{tt}(t) + \sum_{|\alpha|,|\beta|\le m} (-1)^\alpha \partial^\alpha \big(a_{\alpha\beta}(x,t)\partial^\beta u(t)\big) = f(x,t), \quad (x,t)\in Q,\\
&B_j u = 0, \qquad\qquad \text{on } S, j = 1,\cdots,m,\\
&u(x,0) = u_t(x,0) = 0, \qquad\qquad x\in G.
\end{aligned}$$

We study elastic system with structural damping, some of the results achieved in relation to the operators elastic and damping, the properties of semigroup associated elastic system with structural damping, the existence of mild solution, asymptotic behaviour of mild solution were concluded. G. chen and D. L. Russell (1982) obtained some results of relation between elastic operator $\mathcal{A}$ and damping operator $\mathcal{B}$ of system

$$\begin{aligned}
&u_{tt}(t) + \mathcal{B}u_t(t) + \mathcal{A}u(t) = 0, \quad t>0\\
&u(0) = x_0, \quad u_t(0) = y_0.
\end{aligned}$$

A noticeable result from H. Fan, Y. Li and P. Chen (2013) applied semigroup theory of operator and fixed point theorem, and obtained the existence of mild solution in Banach spaces of elastic system with structural damping. In the equation, nonlinear function $f(x,p)$ is Lipschitz in second variable

$$(ESSD) \quad \begin{cases} u_{tt}(t) + \rho\mathcal{A}u_t(t) + \mathcal{A}^2u(t) = f(t,u(t)), \quad 0<t<T,\\ u(0) = x_0, \quad u_t(0) = y_0. \end{cases}$$

H. Fan and F. Gao (2014) discussed Asymptotic behaviour of mild solution of linear elastic system with structural damping and $(ESSD)$.

From all above, we see that there are many problems need to study in initial boundary value problem for nonlinear hyperbolic equations in infinite cylinders with nonsmooth base, in domains with edges and the initial value problem for the differential equations of the second order in the Banach spaces with structural damping, an abstract form of the wave equation structural damping. From this point, we did a futher research on:

- The existence unique of weak solution on $[0,+\infty)$ of initial boundary

value for nonlinear hyperbolic equations of higher order in infinite nonsmooth cylinders.

- The existence unique of weak solution on $[0, T], (0 < T < +\infty)$ of initial boundary value problem for nonlinear hyperbolic equations of second order in some domains with edges.
- The existence of decay mild solution with explicit decay rate of exponential type of nonlocal initial value problem for nonlinear second order differential equation with structural damping in smooth and nonsmooth domains.

3. Purposes, objects and scope of the thesis

We study the existence, uniqueness of global weak solution in weighted Sobolev spaces of initial boundary value problem for semilinear hyperbolic equations in nonsmooth infinite cylinder; Study the existence, uniqueness and regularity of local weak solution of initial boundary value problem for nonlinear hyperbolic in a domain with edges, a cone with edges. Furthermore, we study the existence decay mild solution with explicit decay rate of exponential type of the nonlocal initial value problem with structural damping on the smooth domain and nonsmooth domain with conical point on the boundary in $\mathbb{R}^2$.

4. Research method of the thesis

- In the thesis, we use Galerkin method, embedded theorems, apply fixed point principle, passing through linear problem, help of Dirichlet problem in nonsmooth domains.
- Apply measure of noncompactness, fixed point method for condensing map.

5. Results of thesis

In the thesis, we obtain following main results:

- The existence, uniqueness of weak solution on $[0, +\infty)$ of initial boundary value for semilinear hyperbolic equations higher order in nonsmooth infinite cylinder.
- The existence, uniqueness and regularity of weak solution on $[0, T]$ $(0 < T < +\infty)$ in weighted Sobolev spaces of initial boundary value problem for semilinear hyperbolic equations in a domain with edges and a cone with edges.

- The existence of decay mild solution with explicit decay rate of exponential type of semilinear differential equations second order in Banach spaces with structural damping on nonsmooth and smooth domains in $\mathbb{R}^2$, the values of the nonlinear function in the equation belong to Banach space including continuous bounded functions.

The results of the thesis is a new, scientific significance, and contribute to the improvement of the previous results of the initial boundary value problem for nonlinear hyperbolic equations on a number of specific nonsmooth domains.

6. Structure of the thesis

Besides the introduction, conclusion and list of published works and references, the thesis consists of 4 chapters:
Chapter 1: Preliminaries
Chapter 2: The first initial boundary problem for semilinear hyperbolic equations of higher order in nonsmooth cylinders
Chapter 3: The Dirichlet-Cauchy problem for semilinear hyperbolic in polyhedral domain
Chapter 4: The semilinear wave equation with structural damping.

Chapter 1

PRELIMINARIES

1.1. Spaces of fuctions, weak convergence and Sobolev imbedding theorem

In this Section, some Sobolev spaces are recalled weak convergence and imbedding theorems for Sobolev spaces.

1.2. Some inequalities

Gagliardo-Nirenberg, Cauchy, Young, Hölder, Gronwall inequalities.

1.3. Some basic knowledge of operator theory

1.4. Some embedded lemmas in the domains with edges, Dirichlet problem for elliptic equations in the polyhedral domains

The domains with edges and embedded lemmas. Furthermore, we consider Dirichlet problem for elliptic equations of second order.

1.5. Some embedded lemmas and Dirchlet problems for strong elliptic systems in a cone with edges

The cone with edges and some embedded lemmas. Embedded lemma corresponding to the Dirichlet problem.

1.6. Basic knowledge of semigroup theory of bounded linear operators

In this Section, we recall some basic concepts, properties of semigroup of bounded linear operators and some important semigroups.

1.7. Basis knowledge of measure of noncompactness and condensing map

The concept of measure of noncompactness, in which, Hausdorff measure of non-compact and fixed point principle for condensing maps.

Chapter 2

THE FIRST INITIAL BOUNDARY PROBLEM FOR SEMILINEAR HYPERBOLIC EQUATIONS IN NONSMOOTH CYLINDERS

2.1. Set up the problem

Suppose $\Omega \subset \mathbb{R}^n,\ n \geq 2$ is bounded domain with nonsmooth boundary $\partial\Omega$. Set $Q = \Omega \times (0,\infty), S = \partial\Omega \times (0,\infty)$. We consider the following problem:

$$u_{tt} + (-1)^m Lu + F(x,t,u,Du,\cdots,D^{2m-2}u) = h, \qquad (x,t) \in Q, \tag{2.3}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0, \qquad x \in \Omega, \tag{2.4}$$

$$\frac{\partial^j u}{\partial \nu^j}\Big|_S = 0, \qquad j = 0,1,\cdots,m-1, \tag{2.5}$$

here function $h:\ Q \to \mathbb{R}$ is given, nonlinear function F has form $F = \sum\limits_{0\leq|\alpha|\leq m-1} (-1)^{|\alpha|} D^\alpha F_\alpha(x,t,u,Du,\cdots,D^{m-1}u),\ (x,t) \in Q$, in which there exists function $A = A(x,t,v_\alpha,|\alpha| \leq m-1)$ such that $F_\alpha = \dfrac{\partial A}{\partial v_\alpha}$ is continuous on Q and for all $t \in [0,+\infty),\ x \in \Omega$ following conditions satisfied

$$|F_\alpha(x,t,v_0,v_1,\cdots,v_{m-1})| \leq C\Big(1 + \sum_{|\alpha|\leq m-1} |v_\alpha|^{p+1}\Big), \tag{2.6}$$

$$\begin{aligned} &|F_\alpha(x,t,v_0,v_1,\cdots,v_{m-1}) - F_\alpha(x,t,w_0,w_1,\cdots,w_{m-1})| \\ &\qquad\qquad \leq C \sum_{|\alpha|\leq m-1} \big(|v_\alpha|^p + |w_\alpha|^p\big)|v_\alpha - w_\alpha|; \end{aligned} \tag{2.7}$$

$$\Big|\frac{\partial A}{\partial t}\Big| \leq C\Big(1 + \sum_{|\alpha|\leq m-1} |v_\alpha|^p\Big); \tag{2.8}$$

$$A(x,0,0,\cdots,0) = 0, \quad \int_\Omega A(x,t,u,Du,\cdots,D^{m-1}u)dx \geq 0, \tag{2.9}$$

for a. e. $t \in [0,+\infty)$. Where $\frac{2}{n} \leq p \leq \frac{2}{n-2}$ if $n > 2$ and $1 \leq p < +\infty$ if $n = 2$. Differential operator L of second order $2m$ determined Q with functionial coefficients $a_{\alpha\beta} \in C^1(\overline{Q}),\ a_{\alpha\beta}(x,t) = (-1)^{|\alpha|+|\beta|} a_{\beta\alpha}(x,t),\ |\alpha|,|\beta| = 0,1,\cdots,$

$m,\ (x,t) \in \overline{Q}$. We suppose that there exists positive constant θ such that

$$B[u,u;t] = (-1)^m \int\limits_{\Omega} \sum_{|\alpha|,|\beta| \le m} (-1)^{|\alpha|} a_{\alpha\beta}(x,t) D^\alpha u D^\beta u \ge \theta \|u\|^2_{\mathring{H}^m(\Omega)}$$

hold for $\forall u \in \mathring{H}^m(\Omega),\ \forall\, t \ge 0$. Sobolev space $H^{m,1}_*(Q)$ includes all $u \in L_\infty(0,\infty;\mathring{H}^m(\Omega)), u_t \in L_2(0,\infty;L_2(\Omega)), u_{tt} \in L_2(0,\infty;H^{-m}(\Omega))$,

$$\|u\|_{H^{m,1}_*(Q)} = \|u\|_{L_\infty(0,\infty;\mathring{H}^m(\Omega))} + \|u_t\|_{L_2(0,\infty;L_2(\Omega))} + \|u_{tt}\|_{L_2(0,\infty;H^{-m}(\Omega))}$$

Definition 2.1. Let $h \in L_2\big(0,\infty;L_2(\Omega)\big)$. Then function $u \in H^{m,1}_*(Q)$ is called weak global solution of problem (2.3)-(2.5) iff $u(x,0) = 0, u_t(x,0) = 0,\ x \in \Omega$ and equality

$$\langle u_{tt}(t), v\rangle + B[u(t),v;t] + \sum_{0 \le |\alpha| \le m-1} \big(F(\cdot,t,u,Du,\cdots,D^{m-1}u), D^\alpha v\big) = \big(h(\cdot,t),v\big) \quad (2.10)$$

hold for all $v \in \mathring{H}^m(\Omega)$ and a. e. $t \in [0,+\infty)$. Instead of $t \in [0,\infty)$, considering $t \in [0,T]$ $(0 < T < \infty)$, we call $u \in H^{m,1}_*(Q_T)$ is local weak solution of (2.3)-(2.5).

2.2. Existence and uniqueness of local weak solution

Theorem 2.2. *Let $h \in L_2\big(0,T;L_2(\Omega)\big)$. Then problem (2.3)–(2.5) has local weak solution $u \in H^{m,1}_*(Q_T)$ such that*

$$\|u\|^2_{H^{m,1}_*(Q_T)} \le C\big(1 + \|h\|^2_{L_2(0,T;L_2(\Omega))}\big), \quad (2.32)$$

here C is positive constant independent of u an h.

Theorem 2.3. *Let $h \in L_2\big(0,T;L_2(\Omega)\big)$. Then problem (2.3)–(2.5) has uniqueness local weak solution in $H^{m,1}_*(Q_T)$.*

2.3. Existence and uniqueness of global weak solution

Theorem 2.4. *Let $h \in L_2\big(0,\infty;L_2(\Omega)\big)$. Then problem (2.3)–(2.5) has unique weak global solution in $H^{m,1}_*(Q)$.*

Chapter 3

THE DIRICHLET-CAUCHY PROBLEM FOR NONLINEAR HYPERBOLIC EQUATIONS IN POLYHEDRAL DOMAINS

3.1. The Dirichlet-Cauchy problem for semilinear hyperbolic equations in a domain with edges

3.1.1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n > 2)$ with the boundary $\partial\Omega$ consisting of two surfaces Γ_1, Γ_2 intersect along a manifold l_0. We consider the following problem:

$$u_{tt} + L(x,t,\partial)u = f(x,t,u,Du) + h(x,t), \quad (x,t) \in Q_T, \tag{3.4}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad x \in \Omega \tag{3.5}$$

$$u|_{S_T} = 0, \tag{3.6}$$

where L is differential operator of second order, f, h are given functions.

Definition 3.1. A function $u(x,t)$ is called a weak solution of problem (3.4)–(3.6) on $[0,T]$ iff $u \in L_2\big(0,T;\mathring{H}^1(\Omega)\big)$, $u_t \in L_2\big(0,T;L_2(\Omega)\big)$, $u_{tt} \in L_2\big(0,T;H^{-1}(\Omega)\big)$ such that $u(x,0) = 0$, $u_t(x,0) = 0$, $x \in \Omega$ and the equality

$$\langle u_{tt}(t), v\rangle + B[u(t), v; t] = \big(f(\cdot,t,u,Du), v\big) + \big(h(\cdot,t), v\big)$$

hold for all $v \in \mathring{H}^1(\Omega)$ and for a.e. $t \in [0,T]$. Here, B is time-dependent bilinear form corresponding on L.

3.1.2. The linear problem

a. The existence of solution of the linear problem

If $f(x,t,u,Du) \equiv 0$ on Q_T, we have following linear problem:

$$u_{tt} + L(x,t,\partial)u = h(x,t), \quad (x,t) \in Q_T, \tag{3.7}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad x \in \Omega \tag{3.8}$$

$$u|_{S_T} = 0. \tag{3.9}$$

A function $u(x,t) \in H^1(\Omega)$ is called a weak solution of problem (3.7)–(3.9) on $[0,T]$ if $u \in \mathring{L}_2(0,T;H^1(\Omega)), u_t \in L_2(0,T;L_2(\Omega)), u_{tt} \in L_2(0,T;H^{-1}(\Omega))$,

$u(x,0) = u_t(x,0) = 0, x \in \Omega$ and the equality

$$\langle u_{tt}(t), v\rangle + B[u(t), v; t] = \big(f(\cdot, t), v\big) \tag{3.10}$$

hold for all $v \in \mathring{H}^1(\Omega)$ and for a.e. $t \in [0,T]$.

Theorem 3.1. *Let $h \in C\big(0,T; L_2(\Omega)\big)$. Then problem (3.7)–(3.9) has the unique weak solution $u(t) \in \mathring{H}^1(\Omega), t \in [0,T]$ such that*

$$\|u(t)\|^2_{\mathring{H}^1(\Omega)} + \|u_t(t)\|^2_{L_2(\Omega)} \le C \int\limits_0^t \|f(s)\|^2_{L_2(\Omega)}, \quad \forall t \in (0,T], \tag{3.11}$$

here C is a positive constant depending only on Ω, T and coefficients of L.

b. The regularity in time variable of weak solution of the linear probolem

Theorem 3.2. *Let $k \in \mathbb{N}, h_{t^j} \in C\big([0,T]; L_2(\Omega)\big),\ j = 0, 1, \cdots, k$. Then problem (3.7)–(3.9) has a weak solution u such that*

$$u_{t^j} \in C\big(0,T; \mathring{H}^1(\Omega)\big),\ j = 0, 1, \cdots, k,\ \forall t \in [0,T], \tag{3.39}$$

$$\sum_{j=0}^{k} \|u_{t^j}(t)\|^2_{H^1(\Omega)} + \|u_{t^{k+1}}\|^2_{L_2(\Omega)} \le C \sum_{j=0}^{k} \int\limits_0^t \|h_{t^j}(s)\|^2_{L_2(\Omega)} ds. \tag{3.40}$$

A function u belongs to the space $\mathscr{H}_a^k$, if $u \in \bigcap\limits_{j=0}^{k} C^{(j)}\big(0,T; H_a^{k-j}(\Omega)\big)$. For every $u \in \mathscr{H}_a^k$ and for every $t \in [0,T]$, we define following norms

$$\|u(t)\|^2_{\mathscr{H}_a^k(\Omega)} = \sum_{j=0}^{k} \|u_{t^j}(t)\|^2_{H_a^{k-j}(\Omega)} \text{ and } \|u\|_{\mathscr{H}_a^k} = \sum_{j=0}^{k} \|u_{t^j}\|_{C\left([0,T]; H_a^{k-j}(\Omega)\right)}.$$

Theorem 3.3. *Let $k \in \mathbb{N}, k \ge 2$ and suppose that $h_{t^j} \in C\big(0,T; H_a^{k-j}(\Omega)\big)$, $0 \le j \le k-1,\ f_{t^j}(x,0) = 0, 0 \le j \le k-2$ and $k+1-a < \dfrac{\pi}{\omega},\ a \in [0,1]$. Then weak solution u of the problem (3.7)–(3.9) belongs $\mathscr{H}_a^k$ and*

$$\|u(t)\|^2_{\mathscr{H}_a^k(\Omega)} \le C \int\limits_0^t \|h(s)\|^2_{\mathscr{H}_a^{k-1}(\Omega)} ds,\ t \in (0,T]. \tag{3.56}$$

3.1.3. The semilinear problem

Lemma 3.1. *Let* $v_i \in C^{(k_i)}\big(0,T;H_a^{l-k_i}(\Omega)\big)$, $i=1,\cdots,m$, *where* l, k_i *are* $m+1$ *positive integer such that* $l \geq \dfrac{n}{2}+a+1$, $0 \leq k_1 \leq l-1$, $k_i \leq l-1-\left[\dfrac{n}{2}\right]$, $i=2,\cdots,m$. *Suppose that* $n-$*indices* α_i $(i=1,\cdots,m)$ *such that* $|\alpha_1| \leq l-k_1-1, |\alpha_i| \leq l-k_i-1-\left[\frac{n}{2}\right]$, *và* $\sum\limits_{i=1}^{m}|\alpha_i| \leq l-\sum\limits_{i=1}^{m}k_i$. *Then*

$$\|r^a D^{\nu_1}v_1\cdots D^{\nu_m}v_m\|_{L_2(\Omega)} \leq \prod_{i=1}^{m}\|v_i\|_{H_a^{l-k_i}(\Omega)}. \tag{3.67}$$

The function $f(x,t,v_1,\cdots,v_{n+1})$ is of the class $C^{(k),1}(\overline{Q_T}\times\mathbb{R}^{n+1})$ if $f \in C^{(k)}(\overline{Q_T}\times\mathbb{R}^{n+1})$ and $D^\alpha f(x,t,\cdot)$ is local Lipschitz function in $n+1$ variable $v_1,\cdots,v_{n+1}$, for every $(x,t)\in\overline{Q_T}$, $|\alpha|\leq k$. Let $f \in C^{(k),1}(\overline{Q_T}\times\mathbb{R}^{n+1})$ and for a constant $b \geq 0$, we denote

$$M(b) = \max_{|\alpha|\leq k}\ \sup_{(x,t)\in Q_T}\ \sup_{|v_i|\leq Cb}|D^\alpha f(x,t,v_1,\cdots,v_{n+1})|,$$

where $C>0$ is a given constant.

Lemma 3.2. *Let* $k \geq a+\dfrac{n}{2}$, $f \in C^{(k),1}(\overline{Q}_T\times\mathbb{R}^{n+1})$ *and functions* $u_1,u_2 \in \mathscr{H}_a^{k+1}$ *such that inequality*

$$\max_{t\in[0,T],i\in\{1,2\}}\|u_i(t)\|_{\mathscr{H}_a^{k+1}} \leq b, \tag{3.69}$$

here b *is a positive constant. Set* $F_i(x,t)=f(x,t,u_i,Du_i)$, $i=1,2$. *Then*

(i) $F_i \in \mathscr{H}_a^k$, $i=1,2$ *and there exists a positive constant* C *such that for every* $t\in(0,T)$, *following inequality satisfying*

$$\|F_i(t)\|_{\mathscr{H}_a^k} \leq CM(b)\left(1+\|u_i(t)\|^k_{\mathscr{H}_a^{k+1}}\right),\ i=1,2; \tag{3.70}$$

(ii) There exists a constant $K(b)>0$ *such that for evry* $t\in(0,T)$, *we have*

$$\|F_1(t)-F_2(t)\|_{\mathscr{H}_a^k} \leq K(b)\,\|u_1(t)-u_2(t)\|_{\mathscr{H}_a^{k+1}} \tag{3.71}$$

Theorem 3.4. *Let* $k \geq a+\dfrac{n}{2}, k+1-a<\dfrac{\pi}{\omega}$, $a\in[0,1]$. *Suppose that* $h \in \mathscr{H}_a^{k-1}$, $f \in C^{(k-1),1}(\overline{Q}_\tau\times\mathbb{R}^{n+1})$ *such that* $h_{t^j}(x,0)=0, f_{t^j}(x,0,v,v_1)=$

$0,\ 0 \le j \le k-2$. *Then problem (3.4)–(3.6) has unique solution* $u \in \mathscr{H}_a^k$ *and*

$$\|u\|_{\mathscr{H}_a^k} \le C(1 + \|h\|_{\mathscr{H}^{k-1}}), \tag{3.72}$$

here C *is a constant independent of* u.

3.2. The Dirichlet-Cauchy problem for semilinear hyperbolic equations in cone with edges

3.2.1. Introduction

Let $\mathcal{K} = \left\{x \in \mathbb{R}^3 : \ \frac{x}{|x|} \in \Omega\right\}$ be a bounded cone in $\mathbb{R}^3$ with vertex at the origin. Suppose that the boundary $\partial\mathcal{K}$ consists of the vertex $x = 0$, the edges (half-lines) $M_1, \cdots, M_d$, and smooth (of class C^∞) faces $\Gamma_1, \cdots, \Gamma_d$. This means that $\Omega = \mathcal{K} \cap S^2$ is a domain of polygonal typer on the unit sphere S^2 with sides $\gamma_k = \Gamma_k \cap S^2$. Let $0 < T < \infty$, and $\mathcal{K}_T = \mathcal{K} \times (0,T), \partial\mathcal{K}_T = \bigcup\limits_{i=1}^{d} \Gamma_i \times (0,T)$.

We consider differential operator L and time-dependent bilinear B on $\mathcal{K}_T$ defined as in Subsection 3.1.1. We consider following problem

$$u_{tt}(x,t) + L(x,t,\partial)u = f(x,t,u) + h(x,t), \quad (x,t) \in \mathcal{K}_T, \tag{3.79}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0, \qquad x \in \mathcal{K} \tag{3.80}$$

$$u|_{\partial\mathcal{K}_T} = 0, \tag{3.81}$$

here f and h are given functions.

Definition 3.2. A function $u(x,t)$ is called weak solution of problem (3.79)–(3.81) on $[0,T]$, if $u \in L_2\big(0,T;\mathring{H}^1(\mathcal{K})\big), u_t \in L_2\big(0,T;L_2(\mathcal{K})\big)$, $u_{tt} \in L_2\big(0,T;H^{-1}(\mathcal{K})\big)$ such that $u(x,0) = u_t(x,0) = 0,\ \forall x \in \mathcal{K}$ and equality

$$\big(u_{tt}(t), v\big) + B\big[u,v;t\big] = \big(f(t,u(t)), v\big) + \big(h(t), v\big), \tag{3.82}$$

hold for every $v \in \mathring{H}^1(\mathcal{K})$ and a.e. $t \in [0,T]$.

3.2.2. The linear problem

a. The existence, uniqueness of weak solution of the linear problem

If $f(x,t,u) \equiv 0$ on $\mathcal{K}^P$, we have following problem:

$$u_{tt}(x,t) + L(x,t,\partial)u = h(x,t), \quad (x,t) \in \mathcal{K}_T, \tag{3.83}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad x \in \mathcal{K} \tag{3.84}$$

$$u|_{\mathcal{K}_T} = 0. \tag{3.85}$$

A function $u(x,t)$ is called weak solution of problem (3.83)−(3.85) on $[0,T]$ if $u \in L_2\big(0,T;H^1(\mathcal{K})\big), u_t \in L_2\big(0,T;L_2(\mathcal{K})\big)$, $u_{tt} \in L_2\big(0,T;H^{-1}(\mathcal{K})\big)$ such that $u(x,0)=0, u_t(x,0)=0$ and equality

$$\langle u_{tt}(t), v\rangle + B[u(t), v; t] = \big(h(t), v\big) \tag{3.86}$$

hold for all $v \in \mathring{H}^1(\mathcal{K})$ and for a.e. $t \in [0,T]$.

Theorem 3.5. *For any $h \in C\big(0,T;L_2(\mathcal{K})\big)$ and suppose the coefficients of operator L satisfy*

$$\sup_{1\le i,j\le n} \{|a_{ij}(x,t)|, |b_i(x,t)|, |c(x,t)| : \ (x,t) \in \mathcal{K}_T\} \le \mu, \ \mu = const.$$

Then problem (3.83)−(3.85) has unique weak solution $u(t), t \in [0,T]$ such that

$$\|u(t)\|^2_{\mathring{H}^1(\mathcal{K})} + \|u_t(t)\|^2_{L_2(\mathcal{K})} \le C \int\limits_0^t \|h(s)\|^2_{L_2(\mathcal{K})} ds, \quad \forall t \in (0,T], \tag{3.87}$$

here C is a positive constant only dependent on $\mathcal{K}, T$ and coefficient of operator L.

b. The regularity of local weak solution in time valuable of the linear problem

The following assumptions are used

$(\mathbf{T}_1)$ Suppose that there exists constant $\mu > 0$ such that

$$\sup_{1\le i,j\le n} \{|a_{ijt^s}(x,t)|, |b_{it^s}(x,t)|, |c_{t^s}(x,t)| : \ (x,t) \in \mathcal{K}_T, s = 0,1,2\} \le \mu.$$

$(\mathbf{T_2})$ Suppose $\beta \in \mathbb{R}, \delta = (\delta_1, \cdots, \delta_d) \in \mathbb{R}^d$ such that

$$\beta, \delta_k \in [0,1], \ k = 1, \cdots, d.$$

$(\mathbf{T_3})$ Suppose that the closed strip between the lines $\lambda = -\dfrac{1}{2}$ and $\lambda = \dfrac{1}{2} - \beta$ is free of eigenvalues of the pencil $\mathfrak{U}(\lambda, t), \ t \in [0,T]$, and

$$-\delta_+^{(k)} < \delta_k - 1 < \delta_-^{(k)}, \ k = 1, \cdots, d.$$

Theorem 3.6. *Let* $m \in \mathbb{N}, h_{t^k} \in C\big([0,T]; L_2(\mathcal{K})\big),\ k = 0, 1, \cdots, m$ *and the hypothesis* $(\mathbf{T_1})$ *satisfies. Then problem (3.83)–(3.85) has weak solution* $u(t), t \in [0,T]$ *such that*

$$u_{t^k} \in C\big(0,T; \overset{\circ}{H}{}^1(\mathcal{K})\big),\ k = 0, 1, \cdot, m,\ \forall t \in [0,T], \tag{3.88}$$

$$\sum_{k=0}^{m} \|u_{t^k}(t)\|^2_{\overset{\circ}{H}{}^1(\mathcal{K})} + \|u_{t^{m+1}}\|^2_{L_2(\mathcal{K})} \leq C \sum_{k=0}^{m} \int_0^t \|f_{t^k}(s)\|^2_{L_2(\mathcal{K})} ds. \tag{3.89}$$

We say that a function u belongs space $\mathscr{H}^l_{\beta,\delta}$ if

$$u \in \bigcap_{j=0}^{l} C^{(j)}\big(0,T; V^{l-j}_{\beta,\delta}(\mathcal{K}).$$

For every $u \in \mathscr{H}^l_{\beta,\delta}$ and for any $t \in [0,T]$. We define following norms

$$\|u(t)\|^2_{l,\beta,\delta} = \sum_{j=0}^{l} \|u_{t^j}(t)\|^2_{V^{l-j}_{\beta,\delta}(\mathcal{K})} \text{ và } \|u\|_{\mathscr{H}^l_{\beta,\delta}} = \sum_{j=0}^{k} \|u_{t^j}\|_{C(0,T;V^{l-j}_{\beta,\delta}(\mathcal{K}))}.$$

Theorem 3.7. *Suppose* $(\mathbf{T_1}), (\mathbf{T_2}), (\mathbf{T_3})$ *and following conditions are satisfied*

(i) $h_{t^k} \in V^0_{\beta,\delta}(\mathcal{K}_T),\ k = 0, 1, 2.$

(ii) $h(x, 0) = 0.$

Let $u(t) \in \overset{\circ}{H}{}^1(\mathcal{K}), t \in [0,T]$ *is weak solution of problem (3.83)–(3.85). Then* $u \in \mathscr{H}^2_{\beta,\delta}$ *and*

$$\|u(t)\|^2_{2,\beta,\delta} \leq C \int_0^t \|h(s)\|^2_{1,\beta,\delta} ds,\ t \in [0,T]. \tag{3.90}$$

3.2.3. The semilinear problem

For a function $f : \mathbb{R}^3 \times [0, +\infty) \times \mathbb{R} \to \mathbb{R},\ (x,t,p) \mapsto f(x,t,p)$, we denote

$$f_t(x,t,p) = \frac{\partial f}{\partial t}(x,t,p); f_x(x,t,p) = \frac{\partial f}{\partial x}(x,t,p); f_p(x,t,p) = \frac{\partial f}{\partial p}(x,t,p).$$

For a constant $\alpha \in [0,2]$, we say that function $f(x,t,u)$ satisfies condition $(\mathbf{F})$ if the following conditions are satisfied

$$|f_x(x,t,p)| \le C|p|^{\alpha+1}; \tag{3.100}$$
$$|f_t(x,t,p)| \le C|p|^{\alpha+1}; \tag{3.101}$$
$$|f_p(x,t,p)| \le C(1+|p|^{\alpha}); \tag{3.102}$$
$$|f(x,t,p)-f(x,t,q)| \le C(|p|^{\alpha}+|q|^{\alpha})|p-q|; \tag{3.103}$$
$$|f_x(x,t,p)-f_x(x,t,q)| \le C(|p|^{\alpha}+|q|^{\alpha})|p-q|; \tag{3.104}$$
$$|f_t(x,t,p)-f_t(x,t,q)| \le C(|p|^{\alpha}+|q|^{\alpha})|p-q|; \tag{3.105}$$
$$|f_p(x,t,p)-f_p(x,t,q)| \le C(|p|^{\alpha-1}+|q|^{\alpha-1})|p-q|. \tag{3.106}$$

Lemma 3.3. *Let $\alpha \in [0,2]$. Suppose that $(\mathbf{T_2}),(\mathbf{F})$ are satisfied and $f(x,0,0)=0$. Then for $\forall u_1,u_2 \in \mathscr{H}^2_{\beta,\delta}$ such that*

$$\max_{t\in[0,T] i\in\{1,2\}} \|u_i(t)\|_{\mathscr{H}^2_{\beta,\delta}} \le R.$$

Set $F_i(t)=f(x,t,u_i(t))\,(i=1,2)$, we have

(a) $F_i \in \mathscr{H}^1_{\beta,\delta}$, $i=1,2$ and there exists a constant $C>0$ independent of u_1,u_2 such that for any $t\in[0,T]$, we have

$$\|F_i(t)\|^2_{1,\beta,\delta} \le C\big(\|u_i(t)\|^2_{V^2_{\beta,\delta}(\mathcal{K})}+\|u_i(t)\|^{2(\alpha+1)}_{V^2_{\beta,\delta}(\mathcal{K})}\big),\ i=1,2. \tag{3.108}$$

(b) There exists a constant $C_R>0$ dependent of R such that for every $t\in[0,T]$, we have

$$\|F_1(t)-F_2(t)\|_{1,\beta,\delta} \le C_R\|u_1(t)-u_2(t)\|_{2,\beta,\delta}. \tag{3.109}$$

Theorem 3.8. *Let $\alpha\in[0,2]$. Suppose that $(\mathbf{T_1}),(\mathbf{T_2}),(\mathbf{T_3}),(\mathbf{F})$ are satisfied and*

$$h\in L_2(0,T;\mathscr{H}^1_{\beta,\delta}), h_{t^k}\in V^0_{\beta,\delta}(\mathcal{K}_T),\ k=0,1,2; f(x,0,0)=0, h(x,0)=0.$$

Then problem (3.79)–(3.81) has unique weak solution $u\in\mathscr{H}^2_{\beta,\delta}$ on $[0,\tau], 0<\tau\le T$ and

$$\|u\|_{\mathscr{H}^2_{\beta,\delta}} \le C\big(1+\|h\|^2_{L_2(0,T;\mathscr{H}^1_{\beta,\delta})}\big), \tag{3.127}$$

here C is positive constant independent of u.

Lemma 3.4. *Let $g:\ [0,+\infty)\to[0,+\infty)$ be a nondecreasing function and there exists $\varepsilon>0$ such that*

$$g|_{[0,\varepsilon]}\in C^1([0,\varepsilon]), g(0)=g'(0)=0.$$

Then there exist $0 < \varepsilon_1 < \varepsilon_2$, *dependent on chosen* $b > 0$ *and* ε_1 *is unique satisfying*

$$b + Mg(x) > x, \quad \text{với } 0 \le x < \varepsilon_1, \tag{3.133}$$

$$b + Mg(x) < x, \quad \text{với } \varepsilon_1 < x < \varepsilon_2. \tag{3.134}$$

for b *is sufficiently small (depending on* g*).*

Theorem 3.9. *Let the assumptions in Theorem 3.8 hold. Then problem (3.79)–(3.81) has unique weak solution* $u \in \mathscr{H}^2_{\beta,\delta}$ *on* $[0, T]$ *and*

$$\|u(t)\|_{2,\beta,\delta} \le C\big(1 + \|h\|^2_{L_2(0,T;\mathscr{H}^1_{\beta,\delta})}\big),$$

where C *is a positive constant independent of* u.

Chapter 4

THE SEMILINEAR WAVE EAQUATION WITH TRUCTURAL DAMPING

4.1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space, we consider the following semi-linear wave problem with structural damping involving non-local conditions

$$u_{tt}(t) + \rho\mathcal{A}u_t(t) + \mathcal{A}^2u(t) = f(t, u(t)), \quad t > 0 \tag{4.8}$$

$$u(0) + g(u) = x_0, \quad u_t(0) + h(u) = y_0, \tag{4.9}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \longrightarrow X$ is a closed linear operator, $\rho \geq 2$ is given constant, $x_0 \in \mathcal{D}(\mathcal{A}), y_0 \in X$ and g, h, f are given fuctions.

Definition 4.1. A function $u \in C([0, T]; X)$ is said to be a mild solution of problem (4.8)$-$(4.9) on $[0, T]$ if

$$\begin{aligned} u(t) = S_2(t)\big(x_0 - g(u)\big) &+ \int_0^t S_2(t-s)S_1(s)v_0 ds \\ &+ \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)f\big(\tau, u(\tau)\big)d\tau ds, \end{aligned} \tag{4.16}$$

for any $t \in [0, T]$, where $v_0 = y_0 - h(u) + \sigma_2\mathcal{A}\big(x_0 - g(u)\big)$.

4.2. The existence mild solution of the problem

(A) The operator $-\mathcal{A}$ generates a norm continuous C_0-semigroup $\{T(t)\}_{t\geq 0}$ on Banach space X.

(G) Function $g : \ C\big([0, T]; X\big) \longrightarrow \mathcal{D}(\mathcal{A})$ obeys the following conditions:

(i) g is continuous, and

$$\|g(u)\|_{\mathcal{D}(\mathcal{A})} \leq \theta_g(\|u\|_C), \ \forall u \in C([0, T]; X), \tag{4.17}$$

for all $u \in C([0, T]; X)$, where $\theta_g : \ \mathbb{R} \to \mathbb{R}$ is a position nondecreasing.

(ii) There exist non-negative constants η_g, ξ_g such that

$$\chi(g(\Omega)) \leq \eta_g \, \chi_T(\Omega), , \tag{4.18}$$

$$\chi(\mathcal{A}g(\Omega)) \leq \xi_g \, \chi_T(\Omega), \tag{4.19}$$

for all bounded set $\Omega \subset C([0,T];X)$.

(H) The Fuction $h: \ C([0,T];X) \longrightarrow X$ satisfies following conditions:

(i) There exist a nondecreasing function $\theta_h: \ \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|h(u)\|_X \leq \theta_h(\|u\|_C), \ \forall u \in C([0,T];X), \tag{4.20}$$

(ii) There exists a non-negative η_h such that

$$\chi\big(h(\Omega)\big) \leq \eta_h \, \chi_T(\Omega), \tag{4.21}$$

for all bounded set $\Omega \subset C([0,T];X)$.

(F) Nonlinear function $f: \ \mathbb{R}^+ \times X \longrightarrow X$ satisfies

(i) The nonlinear function $f\big(\cdot, v\big)$ is measurable for each $v \in X, \ f(t,\cdot)$ is continuous for a. e. $t \in [0,T]$ and

$$\|f\big(t,v\big)\|_X \leq m(t)\theta_f(\|v\|_X), \tag{4.22}$$

for all $v \in X, t \in [0,T]$, where $m \in L_1([0,T]), \theta_f: \ \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing function.

(ii) If the semigroup $\{T(t)\}_{t\geq 0}$ is noncompact, there exists function $\eta_f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\eta_f \in L_1(0,T)$ and

$$\chi\big(f(t,\Omega)\big) \leq \eta_f(t) \, \chi_T(\Omega), \tag{4.23}$$

for all bounded set $\Omega \subset X$.

Set

$$S_1(t) = T(\sigma_1 t), S_2(t) = T(\sigma_2 t), \sigma_1 + \sigma_2 = \rho, \sigma_1\sigma_2 = 1, 0 < \sigma_1 < \sigma_2.$$

We denote $B_R = \{u \in C := C([0,T];X): \ \|u\|_C \leq R\}$, where $R > 0$ is given. We see that B_R is bounded convex closed subset in $C([0,T];X)$. We

define solution operator $F:\ B_R \to C$ by

$$F(u)(t) = S_2(t)(x_0 - g(u)) + \int_0^t S_2(t-s)S_1(s)v_0 ds$$
$$+ \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)f\big(\tau, u(\tau)\big)d\tau ds, \quad (4.24)$$

for every $u \in B_R$ and for each $t \in [0,T]$. Set

$$M = \sup_{t\in[0,T]} \|S_2(t)\|_{\mathcal{L}(X)}, \Lambda_T = \sup_{t\in[0,T]} \int_0^t \|S_2(t-s)S_1(s)\|_{\mathcal{L}(X)} ds,$$

$$\Gamma_T = \sup_{t\in[0,T]} \int_0^t \int_0^s \|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)} m(\tau) d\tau ds.$$

$$\Theta_T = \begin{cases} 0, \text{ if the semigroup } \{T(t)\}_{t\geq 0} \text{ is compact,} \\ \sup\limits_{t\in[0,T]} \int\limits_0^t \int\limits_0^s \|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)} \eta_f(\tau) d\tau ds, \text{ otherwise.} \end{cases}$$

Lemma 4.1. *Let* **(A), (G), (H), (F)** *hold. If*

$$\liminf_{n\to\infty} \frac{1}{n}\big[M\theta_g(n) + (\theta_h(n) + \sigma_2\theta_g(n))\Lambda_T + \theta_f(n)\Upsilon_T\big] < 1 \quad (4.25)$$

then there exists $R > 0$ such that $F(B_R) \subset B_R$.

Lemma 4.2. *Let the assumptions of Lemma 4.1 hold. Then*

$$\chi_T\big(F(D)\big) \leq \Big[M\eta_g + 4\big(\eta_h + \sigma_2\xi_g)\Lambda_T + 8\Theta_T\Big]\chi_T(D), \quad (4.31)$$

for all bounded sets $D \subset B_R$.

Theorem 4.1. *Let the assumptions of Lemma 4.2 hold. Then problem (4.8)-(4.9) has at least one mild solution on $[0,T]$ provided that*

$$l := M\eta_g + 4(\eta_h + \sigma_2\xi_g)|\Lambda_T + 8\Theta_T < 1. \quad (4.41)$$

4.3. The existence of decay mild solution of the problem

In this section, we consider solution operator F on the following set:

$$BC_R^\gamma(\beta) = B_R \cap \big\{v \in BC(\mathbb{R}^+; X):\ \sup_{t\in\mathbb{R}^+} e^{\gamma t}\|v(t)\|_X \leq \beta\big\},$$

wher B_R is the closed ball in $BC(\mathbb{R}^+; X)$ centered at origin with radius $R; \beta$ và γ are positive numbers which chosen later. This is a bounded convex closed subset. On $BC_R^\gamma(\beta)$ we make use of MNC χ^*. We will prove that F keeps $BC_R^\gamma(\beta)$. To this end, we have to replace **(A), (G), (H)** and **(F)** by stronger ones:

(A^*) Operator $-\mathcal{A}$ is the infinitesimal generator of a norm-continuous C_0- semigroup $\{T(t)\}_{t\geq 0}$ such that

$$\|T(t)\|_{\mathcal{L}(X)} \leq Ce^{-\delta t},\ t \geq 0,$$

where C, δ are positive constants.

(G^*) The assumption **(G)** is satisfied with any $T > 0$.

(H^*) The assumption **(H)** is satisfied with any $T > 0$.

(F^*) The assumption **(F)** is satisfied with $\theta_f(r) = r,\ \eta_f \in L^\infty(\mathbb{R}^+)$ and $m \in L^1_{loc}(\mathbb{R}^+)$ such that

$$K = \sup_{s\geq 0} \int_0^s e^{-(\delta\sigma_1-\gamma)(s-\tau)} m(\tau)d\tau < +\infty.$$

Set

$$M_\infty = \sup_{t\in\mathbb{R}^+} \|S_2(t)\|_{\mathcal{L}(X)}, \Lambda_\infty = \sup_{t\in\mathbb{R}^+} \int_0^t \|S_2(t-s)S_1(s)\|_{\mathcal{L}(X)} ds,$$

$$\Gamma_\infty = \sup_{t\in\mathbb{R}^+} \int_0^t \int_0^s \|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)} m(\tau)d\tau ds.$$

$$\Theta_\infty = \begin{cases} 0, \text{ if the semigroup } \{T(t)\}_{t\geq 0} \text{ is compact,} \\ \sup\limits_{t\in\mathbb{R}^+} \int\limits_0^t \int\limits_0^s \|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)} \eta_f(\tau)d\tau ds, \text{ otherwise.} \end{cases}$$

In what follows, let $\gamma \in (0, \delta\sigma_1]$ be fixed.

Lemma 4.3. *Let* **(A*), (G*), (H*), (F*)** *hold. If*

$$\liminf_{n\to\infty} \frac{1}{n}\Big[M_\infty\theta_g(n) + \big(\theta_h(n) + \sigma_2\theta_g(n)\big)\Lambda_\infty\Big] + \Gamma_\infty < 1, \tag{4.42}$$

$$\text{and} \quad \frac{KC^2}{\delta\sigma_2 - \gamma} < 1, \tag{4.43}$$

then there exists positive numbers R, β such that $F\big(BC_R^\gamma(\beta)\big) \subset BC_R^\gamma(\beta)$.

Lemma 4.4. *Let* **(A*), (G*), (H*), (F*)** *hold. Then we have*

$$\chi^*\big(F(D)\big) \leq \Big[M_\infty \eta_g + 4\big(\eta_h + \sigma_2 \xi_g\big)\Lambda_\infty + 8\Theta_\infty\Big]\chi^*(D), \tag{4.51}$$

for all bounded sets $D \subset BC_R^\gamma(\beta)$.

Theorem 4.2. *If the assumptions of Lemma 4.3 are satisfied and the inequality*

$$l_\infty := M_\infty \eta_g + 4\big(\eta_h + \sigma_2 \xi_g)\Lambda_\infty\big) + 8\Theta_\infty < 1. \tag{4.59}$$

takes place, then the problem (4.8), (4.9) has at least one mild solution on $\mathbb{R}^+$ *such that* $e^{\gamma t}\|u(t)\|_X = O(1)$ *as* $t \to +\infty$.

CONCLUSION

1. Results of the thesis

- The existence and uniqueness of weak solution on $[0, +\infty)$ of the initial boundary value problem for nonlinear hyperbolic equations of higher order in nonsmooth cylinders. The equation of the problem contains nonlinear function $F(x, t, u, Du, \cdots, D^{2m-2}u)$.
- The existence, uniqueness and asymptotic behaviour on $[0, T]$ $(0 < T < +\infty)$ in time variable in the weighted Sobelev spaces of Dirichlet-Cauchy problem for semilinear hyperbolic equation of second order in domains with edge, cone with edges.
- The existence of decay mild solution with explicit decay rate of exponential type of semilinear wave equations with structural damping involving nonlocal conditions in smooth domains and domain with cone point on the boundary in $\mathbb{R}^2$, the equation of the problem contains for class of nonlinear functions which them value belong Banach space of bounded linear functions.

2. Recommemdation

From the results obtained in the thesis, we found that there are following some issues which need to study:

- Asymptotic behavior of solutions of Dirichlet-Cauchy problem for nonlinear hyperbolic equations in a domain with edges.
- Existence, uniqueness, regularity and asymptotic behavior of solution. Dirichlet-Cauchy problem for nonlinear hyperbolic equations in polyhedral domain
- Existence of decay mild solutions of wave equations with structural damping involving infinite delays and nonlocal conditions.

AUTHOR'S WORKS RELATED TO THE THESIS THAT HAVE BEEN PUBLISHED

The main results were published in 03 articles in international scientific journals prestige (01 article in the ISI category):

1. Vu Trong Luong, Nguyen Thanh Tung, *The first initial boundary value problem for nonlinear hyperbolic equations of higher order in cylinders with singular points*, International journal of evolution equations 2015, Volume 9, Number 2, pp. 167-180

2. Vu Trong Luong, Nguyen Thanh Tung, *The Dirichlet-Cauchy problem for nonlinear hyperbolic equations in a domain with edges*, Nonlinear Analysis 125 (2015) 457-467

3. Vu Trong Luong, Nguyen Thanh Tung, *Decay mild solutions for elastic systems with structural damping involving nonlocal conditions*, Vestnik St. Petersburg University, Mathematics, 2017, Vol. 50, No. 1, pp. 55-67.

Furthermore results of thesis have been reported in:

- Congress of Mathematics Eighth national, Nha Trang, 08/2013;
- Scientific Conference "Mathematical analysis and applications", Hong Duc University, 26-28/5/2016;
- Seminar of the Department of Analysis, Faculty of Informatics mathematics, Hanoi National University of Education.